Symmetric Box-Splines on Root Lattices BIRS Sampling and Reconstruction: Applications and Advances

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Banff rocks!

Overview

- Cowork with Jörg Peters & Alireza Entezari
- Based on the sphere packing problem and sphere covering problem, root lattices are proposed as efficient sampling lattices in arbitrary dimensions.
- Symmetric box-spline filters are constructed for n-dimensional irreducible root lattices, leveraging the symmetric structure of each lattice. (Zⁿ, A_n, A^{*}_n, D_n, D^{*}_n)
- Detailed properties of each box-spline and its spline space are investigated.
- Applications in volume reconstruction are presented.

Minho Kim and Jörg Peters, *Symmetric Box-Splines on Root Lattices*, Journal of Computational and Applied Mathematics (accepted)

Densest Sphere Packing Problem

"How can we arrange non-overlapping identical spheres in the *n*-dimensional Euclidean space maximizing the volume proportion occupied by the spheres?"

- Regular(lattice)/irregular arrangement
- Densest regular packings are known up to dimension 8.



(Courtesy of mathscareers.org.uk)



(Courtesy of old-picture.com)

Densest Regular Packing and Optimal Sampling Lattices



• To find the lattice with maximum density Δ :

 $\Delta = \text{proportion of the space occupied by the spheres}$ $= \frac{\text{volume of the inscribed sphere}}{\text{volume of the Voronoi cell}}$

▶ We want the *inradius* of the Voronoi cells as *large* as possible.

The optimal sampling lattice is the dual of the densest sphere packing lattice (Peteresen and Middleton '62).

Thinnest Sphere Covering



- **•** To find the lattice with minimum thickness Θ:
 - $\Theta = \text{average } \# \text{ of spheres that contain a point in the space}$ $= \frac{\text{volume of the circumsphere}}{\text{volume of the Voronoi cell}}$
- We want the *circumradius* of the Voronoi cells as *small* as possible.

 For dense regular packing or thin regular covering, high symmetry at every lattice point is required.
 → Root lattices

Lattices

- Discrete subgroup of maximal rank in a Euclidean vector space.
- Can be generated by a square generator matrix L.
- Dual lattice can be generated by \mathbf{L}^{-t} .



2-Dimensional Example of Finite Reflection Group

► Why 'reflections'?

 \rightarrow Reflections generate all the rigid transformations.



Finite Reflection Groups

"Which configurations of mirrors result in finite reflection groups in n-dimensional Euclidean space?"

- Answered by H.S.M. Coxeter (1907–2003).
- \blacktriangleright For n=2, dihedral angles π/k , $k\geq 2$ and $k\in\mathbb{Z}$, are allowed.
- For n > 2, only finitely many finite reflection groups exist.
- Symmetric: Invariant under the orthogonal transformations generated by reflections.



Root Systems



- Fundamental roots
- ► A finite reflection group can be re-formulated by a *root system* and studied via linear algebra.

Root Lattices

 Not all root systems generate lattices! (Should be shift-invariant.)



► Crystallographic restriction: Dihedral angles are limited to $\pi/k, \ k \in \{2, 3, 4, 6\}.$



o o c

 Symmetric: Root lattices have the same symmetry as the finite reflection group at every lattice point. Packing Densities of Some Root Lattices (Conway & Sloane)



- ▶ Known to be optimal (among lattices) up to dimension 8:
 ℤ ≃ 𝒜₁, 𝒜₂, 𝒜₃ ≃ 𝒜₃, 𝒜₄, 𝒜₅, 𝔅₆, 𝔅₇, and 𝔅₈
- Cartesian lattices are not efficient sampling lattices.

Covering Thickness (Θ) of Some Root Lattices (Conway & Sloane)



Known to be optimal (among lattices) up to dimension 5:
 Z, A₂, A₃^{*}, A₄^{*}, and A₅^{*}

> Again, Cartesian lattices are not efficient sampling lattices.

- Root lattices are good candidates for efficient sampling in arbitrary dimensions.
- Cartesian lattices are less efficient sampling than other root lattices.
- ► Which (symmetric) reconstruction filter can we use? → Box-splines

Box-Splines: Definition

$m{n} imes m{m}$ Direction matrix $ig[1 \ 1 \ 1]$







y







Box-Splines: Properties

- Finite support defined by Minkowski sum of the directions.
- Piecewise polynomial of degree m n
- Polynomial pieces are delineated by the shifts of the *knot* planes (Hyperplanes spanned by the directions of Ξ).
- Carl De Boor, Klaus Höllig, S. D. Riemenschneider "Box Splines" (1993)

Spline

- ▶ A linear combination of the shifts of the box-spline: $s \in S_{\Xi} := \operatorname{span}(M_{\Xi}(\cdot - j))_{j \in \mathbb{Z}^n}.$
- ▶ $\{M_{\Xi}(\cdot j)\}_{j \in \mathbb{Z}^n}$ form a basis iff Ξ is unimodular.
- ► Approximation order (when $\Xi \in \mathbb{Z}^{n \times m}$) $\rho(\Xi) := \{\min_{\mathbf{Z} \subseteq \Xi} \# \mathbf{Z} : \operatorname{rank}(\Xi \backslash \mathbf{Z}) < n\}$
- Spline evaluation at x.

$$egin{aligned} s(m{x}) &= \sum_{m{j}\in\mathbb{Z}^n} M_\Xi(m{x}-m{j}) a(m{j}) \ &= M_\Xi(m{x}-m{j}_1) a(m{j}_1) \ &+ M_\Xi(m{x}-m{j}_2) a(m{j}_2) \ &+ M_\Xi(m{x}-m{j}_3) a(m{j}_3) \ &+ M_\Xi(m{x}-m{j}_3) a(m{j}_3) \ &+ M_\Xi(m{x}-m{j}_5) a(m{j}_5) \ &+ M_\Xi(m{x}-m{j}_6) a(m{j}_6) \ &+ M_\Xi(m{x}-m{j}_7) a(m{j}_7) \end{aligned}$$



• Large support \rightarrow More samples for evaluation

Box-Splines on Non-Cartesian Lattices

$$\sum_{oldsymbol{j}\in \mathbf{L}\mathbb{Z}^n} |\det \mathbf{L}| M_{\mathbf{L}\Xi}(\cdot-oldsymbol{j}) a(oldsymbol{j}) = \sum_{oldsymbol{k}\in \mathbb{Z}^n} M_{\Xi}(\mathbf{L}^{-1}\cdot-oldsymbol{k}) a(\mathbf{L}oldsymbol{k})$$

A spline as a linear combination of the shifts of the box-spline | det L|M_L = on the non-Cartesian lattice LZⁿ has a change of variables relation with the spline as a linear combination of the shifts of the box-spline M_Ξ on the Cartesian lattice.

Constructing Symmetric Box-Splines on Root Lattices

- To maximize the approximation order, directions associated with the lattice points are used. This also guarantees rational polynomial coefficients. (Kim & Peters '09)
- To make the box-spline has the same symmetry as the lattice, all the (non-parallel) lattice points with the same distances are included.
- To make the support of the box-spline small, consider the short directions first.

Symmetric Box-Spline on the Cartesian Lattice

dim.	box-spine	a ttice	direction matrix	generator matrix	continuity	basis?	$\lambda, (f(\cdot + j))$
n	M_{Z^n}	Girtesiai	$I_n \sqcup \{e_n + \sum_{j=1}^{n-1} \pm e_j\}$	In	C2***	10	tot known
2	$M_{\mathbb{Z}^2}\cong \mathbb{Z}\operatorname{P-e}\mathbf{k} \operatorname{ment}$		$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$		C 1	10	$(f - \frac{1}{24} \sum_{\xi \in \Xi_{2F}} D_{\xi}^{1}f)(j)$
3	M_{2^3}		$\left\{ \begin{array}{rrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrr$		C^{2}	10	$\left(f - \frac{1}{24} \sum_{\xi \in \Xi_{\mathbb{R}^2}} D_{\xi}^1 f\right)(j)$
n	$M^{\pm}_{A_n}$	\mathcal{A}_n	$\bigcup_{1 \le i \le i \le j \le i} \{ \mathbf{X}_n^{\pm} (\mathbf{e}_i - \mathbf{e}_j) \}$	\mathbf{A}_n^{\pm}	C^{n-2}	yes	not known
2	$M^{\pm}_{\mathcal{A}_{2}}\cong M^{\pm}_{\mathcal{A}_{2}}\cong M^{*\pm}_{1}$	bexa go na l	$\frac{1}{2}\begin{bmatrix} 2 & 1 \pm \sqrt{3} & -1 \pm \sqrt{3} \\ -2 & -1 \pm \sqrt{3} & 1 \pm \sqrt{3} \end{bmatrix}$	$\begin{array}{c c} 1 \\ \hline 2 \end{array} \begin{bmatrix} 1 \pm \sqrt{3} & -1 \pm \sqrt{3} \\ -1 \pm \sqrt{3} & 1 \pm \sqrt{3} \end{bmatrix}$	C^1	yes	$f(\mathbf{j})$
3	$M_{A_3} = M_{fee} \cong M_{D_3}$	FCC	$ \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ -1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & 0 \end{bmatrix} $		C 1	yes	$\left(f - \frac{1}{24} \sum_{\xi \in \mathfrak{M}_{i+1}} D_{\xi}^{1}f\right)(j)$
n	$M^{\pm}_{A_n} = M^{s\pm}_1$	\mathcal{A}_n^*	$\mathbf{A}_{n}^{*\pm}$ \mathbf{I}_{n} $-\mathbf{j}$	$\mathbf{A}_n^{\star\pm}$	C^1	yes	f (j)
n	$M_r^{*\pm}$	A_n^*	$\mathbf{A}_{n}^{*\pm} \bigcup \left[\mathbf{I}_{n} - \mathbf{j} \right]$	A _n ^{*±}	C ^{1r-1}	yes	not known
n	$M_1^{\pm\pm}$	A_n^*	$\mathbf{A}_{n}^{*\pm} \mid \mathbf{I}_{n} - \mathbf{j} \mid \mathbf{I}_{n} - \mathbf{j} \mid$	$\mathbf{A}_n^{\pm\pm}$	C^{1}	yes	$(f - \frac{1}{12} \sum_{\xi \in T_{1}^{-1}} D_{\xi}^{1}f)(j)$
2	$M^\pm_{\mathcal{A}_2}\cong M^\pm_{\mathcal{A}_2}\cong M^{*\pm}_1$	hexa go na l	$\frac{1}{2\sqrt{3}} \begin{bmatrix} 1 \pm \sqrt{3} & 1 \mp \sqrt{3} & 2\\ 1 \mp \sqrt{3} & 1 \pm \sqrt{3} & 2 \end{bmatrix}$	$\pm \frac{1}{2\sqrt{3}} \left[\begin{array}{ccc} 1\pm\sqrt{3} & 1\mp\sqrt{3} \\ 1\mp\sqrt{3} & 1\pm\sqrt{3} \end{array} \right]$	C^1	yes	f (j)
3	$M_{\mathcal{A}_{2}}^{\pm}\cong M_{\mathcal{A}_{2}}^{\pm}\cong M_{1}^{*\pm}$	BCC	$\frac{1}{2} \begin{bmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -$	$\frac{1}{2} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$	C^1	yes	f (j)
n	M_{D_n}	\mathcal{D}_n	$\bigcup_{1 \le i \le j} \{ \mathbf{e}_i \pm \mathbf{e}_j \}$	$\begin{bmatrix} I_{n-1} & -e_{n-1} \\ -j^{2} & -1 \end{bmatrix}$	$\begin{cases} C^{1} & (n - 3) \\ C^{2n-4} & (n > 3) \end{cases}$	$\begin{cases} yes & (n - 3) \\ so & (n > 3) \end{cases}$	not known
3	$M_{\mathfrak{D}_2} \simeq M_{\mathcal{A}_2}^+$	FCC	$ \left\{ \begin{array}{cccccccccccccccccccccccccccccccccccc$		C1	ys	$\left(f - \frac{1}{24} \sum_{\xi \in \Xi_{i+1}} D_{\xi}^{1} f\right)(j)$
n	$M_{D_{n}}$	\mathcal{D}_n^*	$\mathbf{I}_n \sqcup \frac{1}{2} \{ \mathbf{e}_n + \sum_{i=1}^{n-1} \pm \mathbf{e}_i \}$	$\begin{bmatrix} I_{n-1} & j/2 \\ 0^t & 1/2 \end{bmatrix}$	C ²ⁿ⁻²	10	not known
3	$M_{\Sigma_{1}}$	BCC	$\frac{1}{2} \left\{ \begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1/2 \end{bmatrix}$	C^{1}	10	$\left(f = \frac{1}{24} \sum_{\xi \in \mathcal{B}_{i,w}} D_{\xi}^{1}f\right)(j)$

Symmetric Box-Spline on the Cartesian Lattice

- The Cartesian lattice \mathbb{Z}^n
 - Generated by the root system $\mathscr{B}_n := \{\pm \mathbf{e}_i \pm \mathbf{e}_j : 1 \le i \ne j \le n\} \cup \bigcup_{1 \le j \le n} \{\mathbf{e}_j\}$
 - Symmetry order: $2^n n!$
 - Center density: 2^{-n}
- (Symmetric) tensor-product B-spline
 - Constructed by the n shortest (axis-aligned) directions, repeated r times each.
 - High degree (rn) compared to its approximation order (r-1).
 - Too large support \rightarrow High computationl cost
- ▶ The symmetric box-spline $M_{\mathbb{Z}^n}$
 - Constructed by n axis-aligned directions + 2ⁿ⁻¹ diagonal directions

 \rightarrow Extension of ZP-element (Zwart '73) and 7-direction box-spline (Peters '96)

- ▶ Polynomial degree: 2ⁿ⁻¹
- Approximation order: $2^{n-2} + 2$
- Shifts do not form a basis.

Box-Spline $M_{\mathbb{Z}^2}$ on the Cartesian Lattice

- $\blacktriangleright \text{ Direction matrix } \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$
- Centered ZP-element (Zwart '73).
- Piecewise polynomial of degree 2.
- C^1 continuous & approximation order 3
- Stencil size is 7.



Box-Spline $M_{\mathbb{Z}^3}$ on the Cartesian Lattice

- $\blacktriangleright \text{ Direction matrix } \left[\begin{array}{rrrrr} 1 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array} \right].$
- Centered 7-direction box-spline (Peters '96).
- Piecewise polynomial of degree 4.
- C^2 continuous & approximation order 4
- Stencil size is 53.

cf. 64 for B-spline with the same approximation order





Symmetric Box-Spline on the \mathcal{A}_n Lattice

dim.	box-spine	a ttice	direction matrix	gererator matrix	continuity	basis?	$\lambda_{i} (f(\cdot + j))$
n 2 3	$M_{Z^n} \label{eq:massed} M_{Z^2} \simeq Z \operatorname{P-ek} \operatorname{ment} M_{Z^2}$	Gartesiaa	$\begin{split} \mathbf{I}_n &\sqcup \{\mathbf{e}_n + \sum_{j=1}^{n-1} \pm \mathbf{e}_j\} \\ & \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ \end{bmatrix} \end{split}$	I,	C ²¹⁻² C ²	10 10 10	$\begin{aligned} & \text{rot krows} \\ & (f - \frac{1}{24} \sum_{\xi \in \Xi_{EF}} D_{\xi}^{1}f)(g) \\ & (f - \frac{1}{24} \sum_{\xi \in \Xi_{EF}} D_{\xi}^{1}f)(g) \end{aligned}$
n	$M^{\pm}_{\mathcal{A}_n}$	\mathcal{A}_n	$\bigcup \{ \mathbf{X}_n^{\pm} (\mathbf{e}_i - \mathbf{e}_j) \}$	\mathbf{A}_n^{\pm}	C^{n-2}	yes	not known
2	$M^\pm_{\mathcal{A}_2}\cong M^\pm_{\mathcal{A}_2}\cong M^{*\pm}_1$	hexa go na l	$\frac{1}{2}\begin{bmatrix} 2 & 1 \pm \sqrt{3} & -1 \pm \sqrt{3} \\ -2 & -1 \pm \sqrt{3} & 1 \pm \sqrt{3} \end{bmatrix}$	$\frac{1}{2} \begin{bmatrix} 1 \pm \sqrt{3} & -1 \pm \sqrt{3} \\ -1 \pm \sqrt{3} & 1 \pm \sqrt{3} \end{bmatrix}$	C^1	yes	f (j)
3	$M_{A_2}^- = M_{tec} \simeq M_{D_2}$	FCC	$ \left[\begin{array}{cccccccc} {}^1\!\!\!& 1 & 1 & 0 & 0 & -1 & -1 \\ -1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & 0 \end{array} \right] $		C^{1}	yes	$\left(f - \frac{1}{24} \sum_{\xi \in \Xi_{i+1}} D_{\xi}^{1}f\right)(j)$
n	$M_{A_{1}}^{\pm} = M_{1}^{s\pm}$	A_n^*	$\mathbf{A}_{n}^{\star\pm} \mid \mathbf{I}_{n} = \mathbf{j}$	A ^{s±}	C^1	yes	f (j)
n	M;**	\mathcal{A}_n^*	$\mathbf{A}_{n}^{*\pm} \bigcup \left[\begin{array}{c} \mathbf{I}_{n} & -\mathbf{j} \end{array} \right]$	$\mathbf{A}_{n}^{s\pm}$	C ^{1r-1}	yes	not known
n	$M_1^{*\pm}$	\mathcal{A}_n^*	$\mathbf{A}_{n}^{*a} \mid \mathbf{I}_{n} - \mathbf{j} \mid \mathbf{I}_{n} - \mathbf{j} \mid$	$\mathbf{A}_n^{\star\pm}$	C^{2}	yes	$(f - \frac{1}{12} \sum_{k=m+1} D_{k}^{1}f)(j)$
2	$M^\pm_{\mathcal{A}_2}\cong M^\pm_{\mathcal{A}_2}\cong M^{\pm\pm}_1$	hexa go na l	$\frac{1}{2\sqrt{3}}\begin{bmatrix} 1 \pm \sqrt{3} & 1 \mp \sqrt{3} & 2\\ 1 \mp \sqrt{3} & 1 \pm \sqrt{3} & 2 \end{bmatrix}$	$\pm \frac{1}{2\sqrt{3}} \begin{bmatrix} 1 \pm \sqrt{3} & 1 \mp \sqrt{3} \\ 1 \mp \sqrt{3} & 1 \pm \sqrt{3} \end{bmatrix}$	C^1	yes	f(j)
3	$M_{A_2}^{\pm} \cong M_{A_2}^{\pm} \cong M_1^{\pm\pm}$	BCC	$\frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -$	$\frac{1}{2} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}^{-1}$	C ¹	ув	$f(\mathbf{i})$
n	M_{D_n}	\mathcal{D}_n	$\bigcup \ \{\mathbf{e}_i \pm \mathbf{e}_j\}$	$\begin{bmatrix} I_{n-1} & -e_{n-1} \\ -\vec{r} & -1 \end{bmatrix}$	$\begin{cases} C^1 & (n = 3) \\ C^{2n-i} & (n > 3) \end{cases}$	$\begin{cases} yes & (n-3) \\ zs & (n > 3) \end{cases}$	not known
3	$M_{\mathcal{D}}, \cong M_{\mathcal{A}}^\pm$	FCC	$ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 \\ \end{pmatrix} $		(° (1 > 1) C1	in (v > n)	$\left(f - \frac{1}{24} \sum_{\xi \in \Xi_{in}} D_{\xi}^{1} f\right)(j)$
n	M_{D_n}	\mathcal{D}_n^*	$\mathbf{I}_n \sqcup \frac{1}{2} \{ \mathbf{e}_n + \sum_{i=1}^{n-1} \pm \mathbf{e}_i \}$	$\begin{bmatrix} I_{n-1} & j/2 \\ 0^t & 1/2 \end{bmatrix}$	C ^{2ⁿ⁻¹}	10	not known
3	$M_{T_{2}}$	BCC	$\frac{1}{2} \begin{bmatrix} 2 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 2 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\left[\begin{array}{ccc} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1/2 \end{array}\right]$	C^1	10	$\left(f = \frac{1}{24} \sum_{\xi \in \Xi_{i,w}} D_{\xi}^{1}f\right)(j)$

Symmetric Box-Spline on the \mathcal{A}_n Lattice

- ▶ The \mathcal{A}_n lattice
 - Generated by the root system
 - $\mathscr{A}_n := \{ \pm (\mathbf{e}_i \mathbf{e}_j) \in \mathbb{R}^{n+1} : 1 \le i \ne j \le n+1 \}$
 - Generated by the vectors associated with the n edges of a regular n-simplex sharing a vertex.
 - Symmetry order: (n + 1)!2
 - Center density: $2^{n/2}(n+1)^{-1/2}$
 - Examples: hexagonal, FCC
- The symmetric box-spline $M^\pm_{\mathcal{A}_n}$
 - ► Constructed by the shortest (non-parallel) n(n + 1)/2 directions.
 - Polynomial degree: n(n-1)/2
 - Approximation order: n
 - The shifts form a basis.

Embedding the \mathcal{A}_n Lattice in \mathbb{R}^n

Diagonally scale the n-simplex composed of

$$\{\mathbf{0}\}\cupigcup_{1\leq j\leq n}\{\mathbf{e}_j\}$$

such that it becomes regular.



Box-Spline $M^\pm_{\mathcal{A}_2}$ on the Hexagonal Lattice

▶ Generator matrix ¹/₂

$$\begin{bmatrix} 1 \pm \sqrt{3} & -1 \pm \sqrt{3} \\ -1 \pm \sqrt{3} & 1 \pm \sqrt{3} \end{bmatrix}$$

 ▶ Direction matrix ¹/₂

$$\begin{bmatrix} 2 & 1 \pm \sqrt{3} & -1 \pm \sqrt{3} \\ -2 & -1 \pm \sqrt{3} & 1 \pm \sqrt{3} \end{bmatrix}$$

Bivariate linear box-spline on the hexagonal lattice.



Box-Spline $M_{
m fcc}$ on the FCC Lattice

Generator matrix
$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$
Direction matrix
$$\begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ -1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & 0 \end{bmatrix}$$

Six-direction box-spline on the FCC lattice (Entezari '07).



Reconstruction on the FCC Lattice

► Quality



Performance

Dataset	Cartesian	FCC	Ratio
Marschner-Lobb	135	98	72%
Carp	515	358	69%

Minho Kim, Alireza Entezari and Jörg Peters, *Box-Spline Reconstruction on the Face Centered Cubic lattice*, IEEE Visualization 2008.

Symmetric Box-Spline on the \mathcal{A}_n^* Lattice

dim.	box-s pine	a ttice	direction matrix	geserator matrix	co sti s sity	basis?	$\lambda, (f(\cdot + j))$
n 2	$M_{\mathbb{Z}^n}$ $M_{\mathbb{Z}^2} \cong \mathbb{Z}\operatorname{P-ek} \operatorname{mest}$	Girtesiai	$\begin{split} \mathbf{I}_n &\sqcup \{\mathbf{e}_n + \sum_{j=1}^{n-1} \pm \mathbf{e}_j\} \\ & \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{split}$	In	C ^{1,-1}	10	tot krow r $(f - \frac{1}{24} \sum_{\xi \in \Xi_{0,r}} D_{\xi}^{1}f)(j)$
3	M_{2^2}		$ \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} $		C^1	10	$\left(f - \frac{1}{24} \sum_{\xi \in \Xi_{\mathbb{R}^2}} D^1_{\xi} f\right)(j)$
n	$M^{\pm}_{A_n}$	\mathcal{A}_n	$\bigcup_{1 \le i \le i \le n+1} {\mathbf{X}_n^{\pm}(\mathbf{e}_i - \mathbf{e}_j)}$	\mathbf{A}_n^\pm	C^{n-2}	yes	tot known
2	$M^\pm_{\mathcal{A}_2}\cong M^\pm_{\mathcal{A}_2}\cong M^{*\pm}_1$	hexa go na l	$\frac{1}{2}\begin{bmatrix} 2 & 1 \pm \sqrt{3} & -1 \pm \sqrt{3} \\ -2 & -1 \pm \sqrt{3} & 1 \pm \sqrt{3} \end{bmatrix}$	$\begin{array}{c} \frac{1}{2} \left[\begin{array}{cc} 1\pm\sqrt{3} & -1\pm\sqrt{3} \\ -1\pm\sqrt{3} & 1\pm\sqrt{3} \end{array} \right] \end{array}$	C^1	yes	f(j)
3	$M_{\mathcal{A}_3}^{-} = M_{tee} \cong M_{\mathcal{D}_3}$	FCC	$ \left[\begin{array}{ccccccccc} 1 & 1 & 0 & 0 & -1 & -1 \\ -1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & 0 \end{array} \right] $		C 1	yes	$\left(f - \frac{1}{24} \sum_{\xi \in \mathfrak{M}_{t+1}} D_{\xi}^{1} f\right)(j)$
n	$M^{\pm}_{\mathcal{A}_n} = M^{*\pm}_1$	A_n^*	$\mathbf{A}_{n}^{*\pm}$ \mathbf{I}_{n} $-\mathbf{j}$	$\mathbf{A}_n^{\star\pm}$	C^1	yes	f (j)
n	M;**	\mathcal{A}_n^*	$\mathbf{A}_{n}^{*\pm} \bigcup \begin{bmatrix} \mathbf{I}_{n} & -\mathbf{j} \end{bmatrix}$	A ^{*±}	C ^{1r-1}	yes	not known
n	$M_1^{s\pm}$	A_n^*	$\mathbf{A}_{n}^{*\pm} \mid \mathbf{I}_{n} - \mathbf{j} \mid \mathbf{I}_{n} - \mathbf{j} \mid$	$\mathbf{A}_{n}^{\star\pm}$	C^{2}	yes	$(f - \frac{1}{12} \sum_{\xi \in T_{1}^{-1}} D_{\xi}^{1}f)(j)$
2	$M^\pm_{\mathcal{A}_2}\cong M^\pm_{\mathcal{A}_2}\cong M^{*\pm}_1$	hexa go na I	$\frac{1}{2\sqrt{3}}\begin{bmatrix} 1 \pm \sqrt{3} & 1 \mp \sqrt{3} & 2\\ 1 \mp \sqrt{3} & 1 \pm \sqrt{3} & 2 \end{bmatrix}$	$\pm \frac{1}{2\sqrt{3}} \left[\begin{array}{ccc} 1\pm\sqrt{3} & 1\mp\sqrt{3} \\ 1\mp\sqrt{3} & 1\pm\sqrt{3} \end{array} \right]$	C^1	yes	f(j)
3	$M_{A_1}^{\pm}\cong M_{A_2}^{\pm}\cong M_1^{\pm\pm}$	BCC	$\frac{1}{2} \begin{bmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -$	$\frac{1}{2} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$	C	yes	f(i)
n	M _{Dn}	\mathcal{D}_n	$[] {\mathbf{e}_i \pm \mathbf{e}_j}$	$\begin{bmatrix} I_{n-1} & -e_{n-1} \end{bmatrix}$	$\begin{cases} C^1 & (n = 3) \\ C^{(n-1)} & (n = 3) \end{cases}$	$\begin{cases} yes & (n-3) \\ (n-3) \end{cases}$	not known
3	$M_{\mathfrak{D}_2}\cong M_{\mathcal{A}_2}^\pm$	FCC	$ \begin{bmatrix} 1 & 0 & \overline{c}_{j \leq n} \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix} $	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$	C1	ую (10 (м.>3)	$\left(f - \frac{1}{24}\sum_{\xi \in \Xi_{lee}} D_{\xi}^{1}f\right)(j)$
n	M_{D_n}	\mathcal{D}_n^*	$\mathbf{I}_n \sqcup \frac{1}{2} \{ \mathbf{e}_n + \sum_{i=1}^{n-1} \pm \mathbf{e}_i \}$	$\begin{bmatrix} I_{n-1} & j/2 \\ 0^t & 1/2 \end{bmatrix}$	C ^{2ⁿ⁻²}	10	not known
3	$M_{T_{2}}$	BCC	$\frac{1}{2} \left[\begin{array}{cccccccccccccccccccccccccccccccccccc$	$\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1/2 \end{bmatrix}$	C^1	10	$(f - \frac{1}{24}\sum_{\xi \in \mathfrak{M}_{i,w}} D^1_{\xi}f)(j)$

Symmetric Box-Spline on the \mathcal{A}_n^* Lattice

- ▶ The \mathcal{A}_n^* lattice
 - The dual lattice of the \mathcal{A}_n lattice
 - Generated by the vectors from the center of a regular n-simplex to its vertices.
 - Symmetry order: (n + 1)!2
 - Center density: $\frac{n^{n/2}}{2^n (n+1)^{(n-1)/2}}$
 - Examples: hexagonal, BCC
- The symmetric box-spline $M_{\mathcal{A}_n^*}^{\pm}$
 - Constructed by the (n + 1) shortest directions.
 - Polynomial degree: 1
 - Approximation order: 2
 - The shifts form a basis.
 - Examples: 4- and 8-direction box-splines on the BCC lattice (Entezari et al.)

Embedding the \mathcal{A}_n^* Lattice in \mathbb{R}^n

Diagonally scale the n-simplex composed of

$$\{-\mathbf{j}\}\cup igcup_{1\leq j\leq n}\{\mathbf{e}_j\}$$

such that it becomes regular.



Symmetric Linear Box-Spline on the \mathcal{A}_n^* Lattice

Analogous to the shadow projection of a slab along diagonal.



Minho Kim, Jörg Peters, Symmetric Box-Splines on the \mathcal{A}_n^* Lattice Journal of Approximation Theory 2010.

Symmetric Box-Spline on the \mathcal{D}_n Lattice

dim.	box-spine	a ttice	direction matrix	generator matrix	continuity	basis?	$\lambda, (f(\cdot + j))$
n 2	$M_{\mathbb{Z}^n}$ $M_{\mathbb{Z}^2} \cong \mathbb{Z}\operatorname{P-ekenest}$	Gartes İa ı	$\begin{split} \mathbf{I}_n &\sqcup \{\mathbf{e}_n + \sum_{j=1}^{n-1} \pm \mathbf{e}_j \} \\ & \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{split}$	In	C11	10 10	not known $(f - \frac{1}{24} \sum_{i \in \Xi_{d_{i}}} D_{i}^{1}f)(j)$
3	M_{2^2}		$\begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$		C^{1}	10	$\left(f - \frac{1}{24} \sum_{\xi \in \mathbf{E}_{22}} D_{\xi}^{1}f\right)(\mathbf{j})$
n	$M^{\pm}_{\mathcal{A}_n}$	\mathcal{A}_n	$\bigcup_{1 \le i \le i \le n+1} {\mathbf{X}_n^{\pm}(\mathbf{e}_i - \mathbf{e}_j)}$	\mathbf{A}_n^\pm	C^{n-2}	yes	not known
2	$M^\pm_{\mathcal{A}_0}\cong M^\pm_{\mathcal{A}_0}\cong M^{*\pm}_1$	hexa go na l	$\frac{1}{2}\begin{bmatrix} 2 & 1 \pm \sqrt{3} & -1 \pm \sqrt{3} \\ -2 & -1 \pm \sqrt{3} & 1 \pm \sqrt{3} \end{bmatrix}$	$\begin{array}{c} \frac{1}{2} \left[\begin{array}{cc} 1\pm\sqrt{3} & -1\pm\sqrt{3} \\ -1\pm\sqrt{3} & 1\pm\sqrt{3} \end{array} \right] \end{array}$	C^1	yes	f(j)
3	$M_{A_3}^ M_{trr} \cong M_{D_3}$	FCC	$ \left[\begin{array}{ccccccccc} 1 & 1 & 0 & 0 & -1 & -1 \\ -1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & -1 & -1 & -1 & 0 \end{array} \right] $		C^{1}	yes	$\left(f - \frac{1}{24} \sum_{\xi \in \mathfrak{M}_{tot}} D_{\xi}^{1} f\right)(j)$
n	$M_{A_n}^{\pm} = M_1^{\pm\pm}$	\mathcal{A}_n^*	$\mathbf{A}_{n}^{*\pm} \mid \mathbf{I}_{n} = \mathbf{j}$	$\mathbf{A}_n^{\star\pm}$	C^1	yes	f (j)
n	$M_{\tau}^{s\pm}$	\mathcal{A}_n^*	$A_n^{*\pm} \bigcup I_n = j$	A ^{*±}	C ^{1r =1}	yes	not known
n	$M_1^{\pm\pm}$	A_n^*	$\mathbf{A}_{n}^{*\pm} \mid \mathbf{I}_{n} - \mathbf{j} \mid \mathbf{I}_{n} - \mathbf{j} \mid$	$\mathbf{A}_{n}^{\star\pm}$	C^{1}	yes	$(f - \frac{1}{12} \sum_{\xi \in T_{i}^{-1}} D_{\xi}^{1}f)(j)$
2	$M^\pm_{\mathcal{A}_2}\cong M^\pm_{\mathcal{A}_2}\cong M^{*\pm}_1$	hexa go na l	$\frac{1}{2\sqrt{3}} \begin{bmatrix} 1 \pm \sqrt{3} & 1 \mp \sqrt{3} & 2\\ 1 \mp \sqrt{3} & 1 \pm \sqrt{3} & 2 \end{bmatrix}$	$\pm \frac{1}{2\sqrt{3}} \left[\begin{array}{cc} 1\pm\sqrt{3} & 1\mp\sqrt{3} \\ 1\mp\sqrt{3} & 1\pm\sqrt{3} \end{array} \right]$	C^1	yes	f (i)
3	$M_{\mathcal{A}_{2}}^{\pm} \cong M_{\mathcal{A}_{2}}^{\pm} \cong M_{1}^{*\pm}$	BCC	$\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -$	$\frac{1}{2}$ $\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$	C^1	yes	f (j)
n	M_{D_n}	\mathcal{D}_n	$\bigcup \ \{ \mathbf{e}_i \pm \mathbf{e}_j \}$	$\begin{bmatrix} I_{n-1} & -e_{n-1} \\ -\vec{r} & -1 \end{bmatrix}$	$\begin{cases} C^1 & (n = 3) \\ C^{2n-4} & (n > 3) \end{cases}$	$\begin{cases} yes & (n-3) \\ 10 & (n > 3) \end{cases}$	not known
3	$M_{\mathfrak{D}_2}\cong M_{\mathcal{A}_2}^\pm$	FCC	$ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix} $	$ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & -1 \end{bmatrix} $	С ¹	ys ($\left(f - \frac{1}{24} \sum_{\xi \in \mathcal{B}_{ini}} D_{\xi}^{1}f\right)(j)$
n	$M_{\Sigma_{n}}$	\mathcal{D}_n^*	$I_n \sqcup \frac{1}{2} \{ e_n + \sum_{j=1}^{n-1} \pm e_j \}$	$\begin{bmatrix} I_{n-1} & j/2 \\ 0^\ell & 1/2 \end{bmatrix}$	C ²ⁿ⁻¹	10	not known
3	$M_{D_{2}}$	BCC	$\frac{1}{2} \left[\begin{array}{cccccccccc} 2 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 2 & 1 & 1 & 1 & 1 \end{array} \right]$	$\left[\begin{array}{ccc} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1/2 \end{array}\right]$	C^{1}	10	$\left(f = \frac{1}{24} \sum_{\xi \in \mathfrak{M}_{t=}} D^1_{\xi} f\right)(j)$

Symmetric Box-Spline on the \mathcal{D}_n Lattice

▶ The \mathcal{D}_n lattice

- ightarrow A.k.a. "checkerboard lattice" $\{oldsymbol{j}\in\mathbb{Z}^n:\sum_koldsymbol{j}(k) ext{ is even}\}$
- Generated by the root system \mathscr{C}_n or \mathscr{D}_n
- Defined only for $n \ge 3$
- Symmetry order: $\begin{cases} 2^n n! & (n \neq 4) \\ 1152 & (n = 4) \end{cases}$
- Center density: $2^{(n+2)/2}$
- Example: FCC

• The symmetric box-spline $M_{\mathcal{D}_n}$

- Constructed by the n(n-1) shortest directions
- Polynomial degree: n(n-2)
- Approximation order: 2n 2
- The shifts do not form a basis except for n = 3.
- Example: 6-direction box-splines on the FCC lattice

Symmetric Box-Spline on the \mathcal{D}_n^* Lattice

dim.	box-spine	a ttice	direction matrix	gererator matrix	continuity	basis?	$\lambda_{i} (f(\cdot + j))$
n 2 3	$M_{\mathbb{Z}^n}$ $M_{\mathbb{Z}^2} \cong \mathbb{Z} \operatorname{P-ekmest} M_{\mathbb{Z}^2}$	Gintesian	$\begin{split} \mathbf{I}_n \cup \{\mathbf{e}_n + \sum_{j=1}^{n-1} \pm \mathbf{e}_j\} \\ & \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \end{split}$	In	C ^{1,,} C ¹ C ²	10 10	rot krows $(f - \frac{1}{24} \sum_{\xi \in \Xi_{g_F}} D_{\xi}^1 f)(j)$ $(f - \frac{1}{24} \sum_{\xi \in \Xi_{g_F}} D_{\xi}^1 f)(j)$
n 2 3	$M^{\pm}_{A_1}$ $M^{\pm}_{A_2} \simeq M^{\pm}_{A_2} \simeq M^{*\pm}_1$ $M^{-}_{A_3} - M_{tet} \simeq M_{Te}$	А _л texagonal FCC	$\begin{array}{c} \bigcup\limits_{1 \le i \le j > i+1} \{ \mathbf{X}_{+}^{*}(\mathbf{e}_{i} - \mathbf{e}_{j}) \} \\ \frac{1}{2} \begin{bmatrix} 2 & 1 \pm \sqrt{3} & -1 \pm \sqrt{3} \\ -2 & -1 \pm \sqrt{3} & 1 \pm \sqrt{3} \\ 1 & 1 & 0 & 0 & -1 & -1 \\ -1 & 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & -1 & -1 & 0 \end{bmatrix}$	$\begin{array}{c} \mathbf{A}_{n}^{\pm} \\ \frac{1}{2} \begin{bmatrix} 1\pm\sqrt{3} & -1\pm\sqrt{3} \\ -1\pm\sqrt{3} & 1\pm\sqrt{3} \end{bmatrix} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	C ⁿ⁻² C ¹ C ²	ys ys ys	rot krow r $f(\boldsymbol{j})$ $(f - \frac{1}{24} \sum_{\boldsymbol{\xi} \in \boldsymbol{\Xi}_{tes}} D_{\boldsymbol{\xi}}^{1} f)(\boldsymbol{j})$
n n 2 3	$\begin{split} M^\pm_{\mathcal{A}_1} &= M^{\pm\pm}_1 \\ M^{\pm\pm}_1 \\ M^{\pm\pm}_1 \\ M^{\pm\pm}_{\mathcal{A}_2} &\simeq M^{\pm\pm}_{\mathcal{A}_2} \simeq M^{\pm\pm}_1 \\ M^{\pm}_{\mathcal{A}_2} &\simeq M^{\pm\pm}_2 \\ M^{\pm\pm}_{\mathcal{A}_2} &\simeq M^{\pm\pm}_2 \\ \end{split}$	A's A's A's hexagonal BCC	$ \begin{array}{c} \mathbf{A}_{a}^{a,b} \left[\mathbf{I}_{a} - \mathbf{j} \right] \\ \mathbf{A}_{a}^{a,b} \left[\mathbf{J}_{a} - \mathbf{j} \right] \\ \mathbf{A}_{a}^{a,b} \left[\mathbf{I}_{a} - \mathbf{j} \right] \\ \mathbf{A}_{a}^{a,b} \left[\mathbf{I}_{a} - \mathbf{j} \mathbf{I}_{a} - \mathbf{j} \right] \\ \mathbf{J}_{a}^{a,b} \left[\mathbf{I}_{a} + \sqrt{\mathbf{J}} \mathbf{I}_{a} + \sqrt{\mathbf{J}} \mathbf{I}_{a}^{2} \\ \mathbf{J}_{a}^{2} \left[\mathbf{I}_{a} + \sqrt{\mathbf{J}} \mathbf{I}_{a}^{2} + \sqrt{\mathbf{J}} \mathbf{I}_{a}^{2} \\ \mathbf{I}_{a} - \mathbf{I}_{a} \mathbf{I}_{a} - \mathbf{I}_{a} \mathbf{I}_{a} - \mathbf{I}_{a} \right] \\ \end{array} \right] $	$\begin{array}{c} \mathbf{A}_{n}^{*n} \\ \mathbf{A}_{n}^{*n} \\ \mathbf{A}_{n}^{*n} \\ \mathbf{A}_{n}^{*n} \\ \pm \frac{1}{2\sqrt{3}} \begin{bmatrix} 1 \pm \sqrt{3} & 1 \pm \sqrt{3} \\ 1 \pm \sqrt{3} & 1 \pm \sqrt{3} \\ 1 \pm \sqrt{3} & 1 \pm \sqrt{3} \\ 1 \end{bmatrix} \\ \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \end{array}$	C ¹ C ² - 2 C ¹ C ¹	ув ув ув ув	$\begin{split} f(j) & \\ & \text{tot known} \\ (f - \frac{1}{12} \sum_{\substack{q \in \mathbb{T}_2^{+k} \\ f(q)}} D_q^1 f)(q) \\ & f(j) \\ \end{split}$
n 3	$M_{\mathcal{D}_n}$ $M_{\mathcal{D}_2} \cong M_{\mathcal{A}_2}^{\pm}$	Dn FCC	$\left. \begin{array}{c} \bigcup\limits_{1 \leq i < j \leq n} \{ \mathbf{e}_i \pm \mathbf{e}_j \} \\ \left[\begin{array}{cccc} 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array} \right] \end{array} \right.$	$\begin{bmatrix} I_{n-1} & -e_{n-1} \\ -\vec{j}^T & -1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$	$\begin{cases} C^1 & (n = 3) \\ C^{2n-4} & (n > 3) \end{cases}$ C^2	$\begin{cases} \text{yes} & (n = 3) \\ \text{so} & (n > 3) \\ \text{yes} \end{cases}$	rot knows $(f - \frac{1}{24} \sum_{g \in \Xi_{t+s}} D_g^2 f)(g)$
n 3	$M_{D_{n}}$ $M_{D_{2}}$	D _n BCC	$\begin{split} \mathbf{I}_n &\sqcup \frac{1}{2} \{ \mathbf{e}_n + \sum_{j=1}^{n-1} \pm \mathbf{e}_j \} \\ \frac{1}{2} \begin{bmatrix} 2 \ 0 \ 0 \ 1 \ -1 \ 1 \ -1 \\ 0 \ 2 \ 0 \ 1 \ 1 \ -1 \ -1 \\ 0 \ 0 \ 2 \ 1 \ 1 \ 1 \ 1 \end{bmatrix} \end{split}$	$\left[\begin{array}{ccc} \mathbf{I}_{n-1} & j/2 \\ 0^{d} & 1/2 \\ 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1/2 \end{array} \right]$	C ²	10	rot krown $(f = \frac{1}{24} \sum_{\substack{q \in \mathbf{Z}_{n-1}}} D_q^1 f)(j)$

Symmetric Box-Spline on the \mathcal{D}_n^* Lattice

▶ The \mathcal{D}_n^* lattice

- The dual lattice of the \mathcal{D}_n lattice
- Generated by inserting additional points at the center of the cubes embedded in Zⁿ. → "Body-centered cubic lattice"
- Symmetric order: same as that of the \mathcal{D}_n lattice

• Center density:
$$\begin{cases} 3^{1.5}2^{-5} & (n=3) \\ 2^{-(n-1)} & (n>3) \end{cases}$$

Example: BCC

• The symmetric box-spline $M_{\mathcal{D}_n^*}$

- ▶ Polynomial degree: 2ⁿ⁻¹
- Approximation order: $2^{n-2} + 2$
- The shifts do not form a basis.

Box-Spline $M_{\mathcal{D}_3^*}$ on the BCC Lattice

• Direction matrix
$$\frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 2 & 1 & 1 & 1 & 1 \end{bmatrix}$$
.
• Generator matrix $\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.



$M_{\mathcal{D}_3^*}$ vs. 8-Direction Box-Spline (Entezari *et al.* '04)

box-spline	8-dir.	$M_{\mathcal{D}_3^*}$
polynomial degree	5	4
approximation order	3	3
<pre># of pieces</pre>	192	720
stencil size	32	30
basis?	yes	no

$M_{\mathcal{D}_3^*}$ vs. 8-Direction Box-Spline (Entezari *et al.* '04)



Wrap-Up

- ▶ Root lattices ⇒ Good!
- ▶ Box-splines ⇒ Good!
- ▶ Box-splines on root lattices ⇒ Awesome!!!

WANTED: Applications in high dimensions

Questions?

