Fast and Stable Evaluation of Box-Splines via the BB-Form

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Motivation

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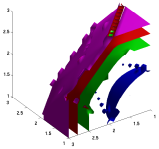

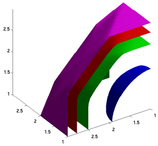

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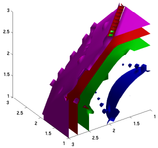

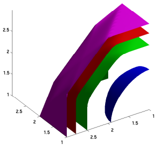

- Existing methods via BB-form (Chui et al. ’91 and Casciola et al. ’06) evaluate only specific box-splines.

Box-spline

$$M_{\Xi}$$

Box-spline

M_{Ξ}

direction matrix

Example

$$\Xi = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

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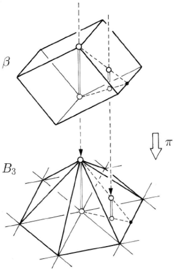
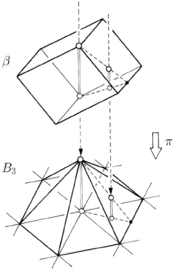


(image courtesy of Prautzsch et al.)

Support

$$\Xi = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$M_{\Xi}$$



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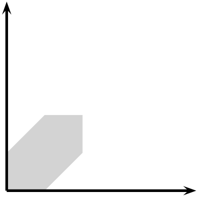

Knot planes

$$M_{\Xi}$$

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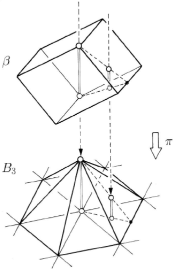

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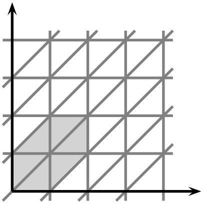
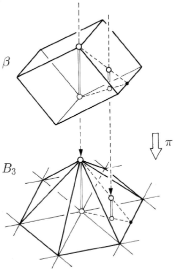
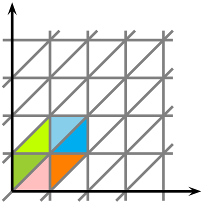


Piecewise polynomial

$$\Xi = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$M_{\Xi}$$





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Spline

$$\begin{array}{c} M_{\Xi} \\ \downarrow \\ * \\ \uparrow \\ a \end{array} \longrightarrow \sum_{j \in \mathbb{Z}^s} a(j) M_{\Xi}(\cdot - j)$$

Evaluation methods

	approximate	exact
M_{Ξ}		

Evaluation methods

	approximate	exact
M_{Ξ}	▶ subdivision	

Evaluation methods

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Evaluation methods

M_Ξ

approximate

- subdivision
- sampling & interpolation
- inverse FFT

exact

- recursive
  de Boor ’93
  Kobbelt ’97

Evaluation methods

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M_Ξ	▶ subdivision ▶ sampling & interpolation ▶ inverse FFT	▶ recursive de Boor '93 Kobbelt '97 ▶ BB-form

Conversion to BB-form

$$M_{\Xi} = \sum c_{\alpha} b_{\alpha} \left(\beta_{\sigma} \left(\cdot \right) \right)$$

Conversion to BB-form

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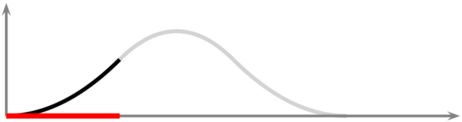
barycentric coordinate function
w.r.t. the domain simplex σ

Conversion to BB-form

$$M_{\Xi} = \sum c_{\alpha} b_{\alpha} (\beta_{\sigma} (\cdot))$$

Bernstein basis polynomial

Conversion to BB-form



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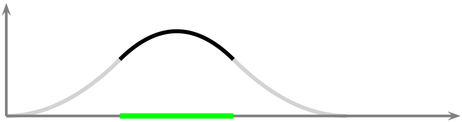

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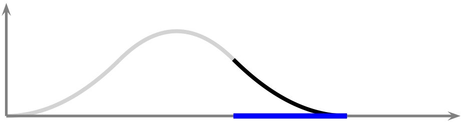

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How to compute the coefficients?

Conversion to BB-form

$$M_{\Xi} = \sum c_\alpha b_\alpha \left(\beta_{\sigma}(\cdot)\right)$$

How to index a polynomial piece (domain simplex σ)?

Computing coefficients

M_{Ξ} $\qquad$ c_{α}

Computing coefficients

$$M_{\Xi}(x_i) = \sum c_\alpha b_\alpha \left(\beta_\sigma \left(x_i\right)\right)$$

sample points

sample points

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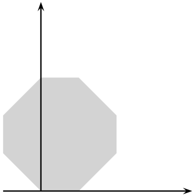
Theorem

Let $\Xi \in \mathbb{Z}^{s\times n}$ and $\operatorname{rank}(\Xi) = s$. Then the polynomial pieces of M_Ξ can be represented in BB-form with coefficients in $\mathbb{Q}$.

Indexing polynomial piece (domain simplex)

$$\Xi = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

M_{Ξ}



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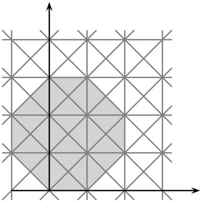

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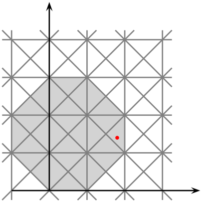

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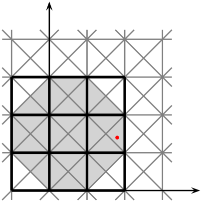

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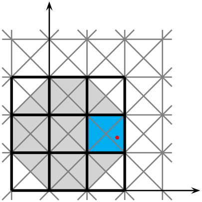

$\mathbf{i}_\Box$

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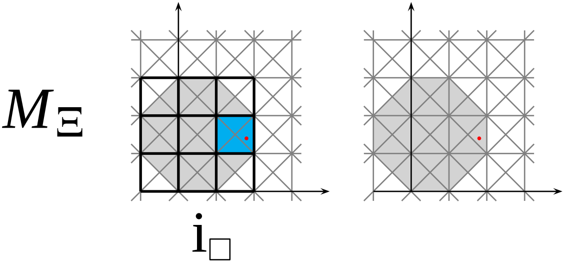

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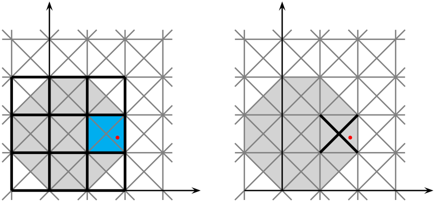

$\mathbf{i}_{\square}$

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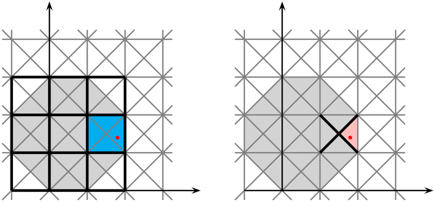

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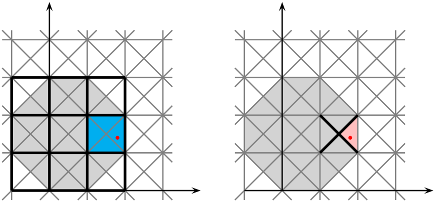

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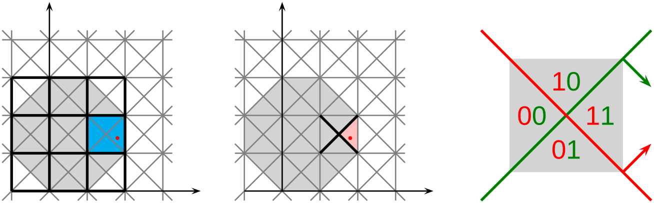

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Spline evaluation

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$\textsc{EvaluateSpline}_{\Xi}(a, x)$

$i_{\triangle} \leftarrow U\left(N_{\Xi}\left(x - \lfloor x \rfloor\right) - \eta_{\Xi}\right)$

$u \leftarrow \textsc{ComputeBarycentric}(i_{\triangle}, x - \lfloor x \rfloor)$

$P \leftarrow \sum_{i_{\square} \in \mathbf{I}_{\Xi}} a(\lfloor x \rfloor - i_{\square}) C_{\Xi}(i_{\square}, i_{\triangle})$

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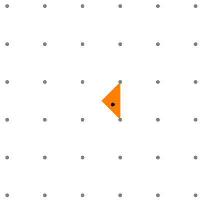

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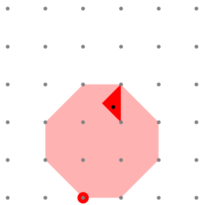

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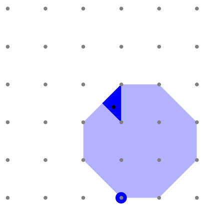

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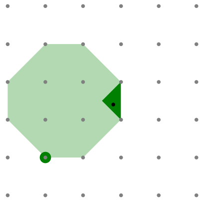

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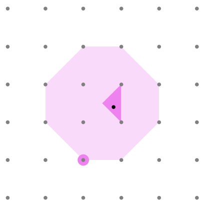

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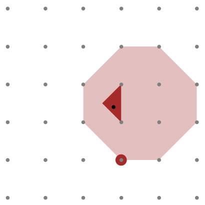

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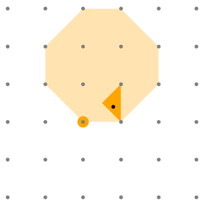

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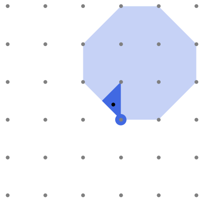

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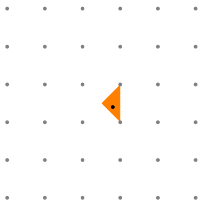

6-directional tri-variate box-spline M_{Ξ_6} on the FCC lattice

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- defined by the direction matrix

$$\Xi_6 := \begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

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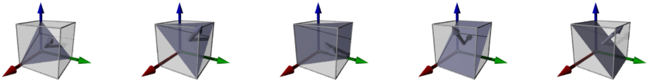
- piecewise polynomial of degree ≤ 3
- equivalent to $M_{\tilde{\Xi}_6}$ on the Cartesian lattice with

$$\tilde{\Xi}_6 := \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

6-directional tri-variate box-spline M_{Ξ_6} on the FCC lattice (cont'd)

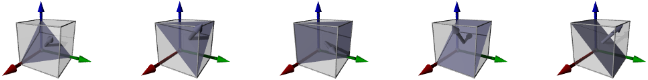
6-directional tri-variate box-spline M_{Ξ_6} on the FCC lattice (cont'd)

- knot planes of $M_{\tilde{\Xi}_6}$ of in $[0..1)^3$

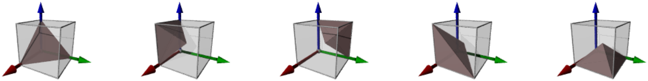


6-directional tri-variate box-spline M_{Ξ_6} on the FCC lattice (cont'd)

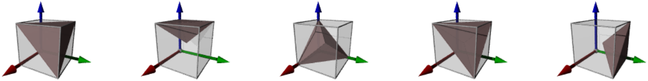
- knot planes of $M_{\tilde{\Xi}_6}$ of in $[0..1)^3$



- polynomial pieces (domain tetrahedra) of $M_{\tilde{\Xi}_6}$ in $[0..1)^3$

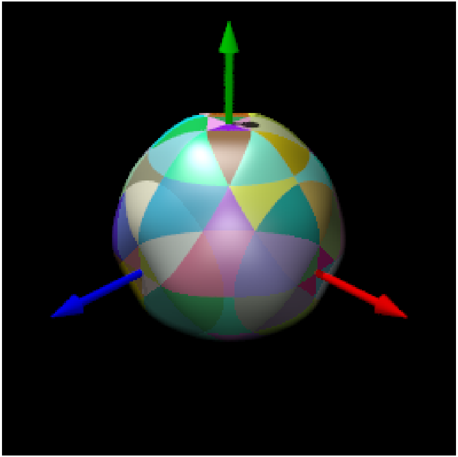


00000 10010 10001 10110 10100



11111 10011 10000 10111 10101

6-directional tri-variate box-spline M_{Ξ_6} on the FCC lattice (cont'd)



6-directional tri-variate box-spline M_{Ξ_6} on the FCC lattice (cont'd)

7-directional tri-variate box-spline M_{Ξ_7}

7-directional tri-variate box-spline M_{Ξ_7}

- defined by the direction matrix

$$\Xi_7 := \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

7-directional tri-variate box-spline M_{Ξ_7}

- defined by the direction matrix

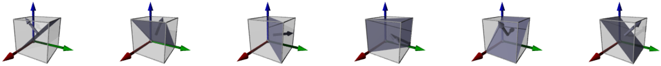
$$\Xi_7 := \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & -1 & 1 \end{bmatrix}$$

- piecewise polynomial of degree ≤ 4

7-directional tri-variate box-spline M_{Ξ_7} (cont'd)

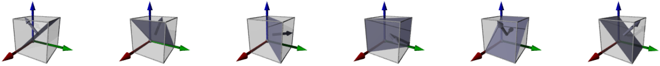
7-directional tri-variate box-spline M_{Ξ_7} (cont'd)

- knot planes in $[0..1)^3$

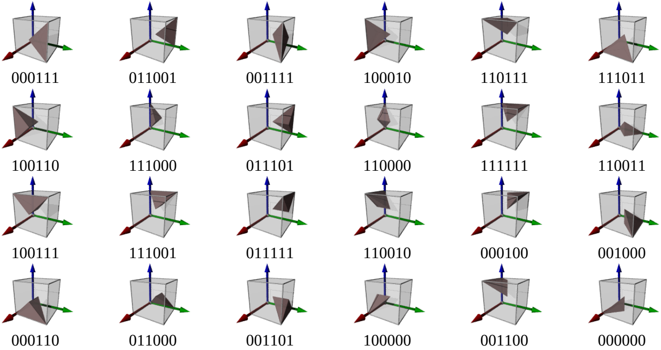


7-directional tri-variate box-spline M_{Ξ_7} (cont'd)

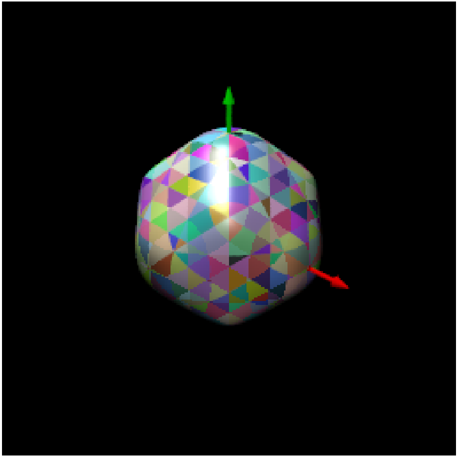
- knot planes in $[0..1)^3$



- polynomial pieces (domain tetrahedra) in $[0..1)^3$



7-directional tri-variate box-spline M_{Ξ_7} (cont'd)



7-directional tri-variate box-spline M_{Ξ_7} (cont'd)

Performance

algorithm	spline	resolution 21^3	31^3	41^3
de Boor	M_{Ξ_7}	20.273238 ×**144**	75.297004 ×**154**	187.711522 ×**153**
	M_{Ξ_6}	1.860688 ×**34**	7.087524 ×**39**	18.147211 ×**41**
Kobbelt	M_{Ξ_7}	52.727976 ×**375**	207.840594 ×**424**	550.422698 ×**450**
	M_{Ξ_6}	3.644995 ×**66**	14.034635 ×**78**	37.232097 ×**84**
via BB-form	M_{Ξ_7}	0.140722	0.489674	1.223360
	M_{Ξ_6}	0.055346	0.180976	0.444804

(evaluation of vectorized input by MATLAB®)

- time measured in secs
- BB-form method is ×ratio times faster

High-quality image generation using ray-tracer

Thank you!

- The MATLAB® package can be downloaded at `http://www.cise.ufl.edu/research/SurfLab/tribox`

References

- Carl de Boor, *On the evaluation of box splines*, Numerical Algorithms **5** (1993), no. 1–4, 5–23.
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- Minho Kim and Jörg Peters, *Fast and stable evaluation of box-splines via the Bézier form*, Tech. Report REP-2007-422, University of Florida, 2007.
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- Hartmut Prautzsch and Wolfgang Boehm, *The handbook of computer aided geometric design*, 3rd ed., ch. Box Splines, pp. 255–282, Elsevier, Amsterdam, 2002.

Spline on non-Cartesian lattice

A spline can also be generated on the non-Cartesian lattice $X^{-1}\mathbb{Z}^s$ spanned by M_Ξ with the coefficients $b : X^{-1}\mathbb{Z}^s \to \mathbb{R}$ (change of variables):

$$\sum_{j \in X^{-1}\mathbb{Z}^s} M_\Xi(\cdot - j)\, b(j) = |\det X| \sum_{j \in \mathbb{Z}^s} M_{X\Xi}(X \cdot - j)\, b(X^{-1} j).$$